Statistical physics

lecture 10

Szymon Stoma

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Conditional probability is the probability of some event A, given the occurrence of some other event B. Conditional probability is written $P(A \mid B)$, and is read "the (conditional) probability of A, given B" or "the probability of A under the condition B". Given a probability space and two events A, B with $P(B) > 0$, the conditional probability of A given B is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Definition

Two events A and B are independent $\Leftrightarrow P(A \cap B) = P(A)P(B)$. Here $A \cap B$ is the intersection of A and B which is the event that both events A and B occur.

If two events A and B are independent, then the conditional probability of A given B is the same as the unconditional probability of A that is $P(A \mid B) = P(A)$.

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A Markov chain is a sequence of random variables $X_1, X_2, X_3, \ldots$ with the Markov property, namely that, given the present state, the future and past states are independent. Formally:

$$P(X_{n+1} = x | X_1 = x_1, X_2 = x_2 \ldots, X_n = x_n) = P(X_{n+1} = x | X_n = x_n).$$

The possible values of X_i form a countable set S called the state space of the chain.

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Let's assume that the state space is finite $S = \{x_1, ..., x_r\}$. Then we can write P in a form of transition matrix:

$$P = \begin{bmatrix} P(x_1,x_1) & P(x_1,x_2) & ... & P(x_1,x_r) \\ P(x_2,x_1) & P(x_2,x_2) & ... & ... \\ ... & & & ... \\ P(x_r,x_r) & ... & ... & P(x_r,x_r) \end{bmatrix}$$

Where $P(x_i, x_j)$ denotes the probability of transition from x_i to x_j. Notice that each row represents all the transition probabilities out of a given state. These probabilities sum up to 1. This transition matrix is a stochastic matrix.

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The probabilities of weather conditions, given the weather on the preceding day, can be represented by a transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

The matrix P represents the weather model in which a sunny day is 90% likely to be followed by another sunny day, and a rainy day is 50% likely to be followed by another rainy day. The columns can be labelled "sunny" and "rainy" respectively, and the rows can be labelled in the same order.

$p_{i,j}$ is the probability that, if a given day is of type i, it will be followed by a day of type j.

Notice that the rows of P sum to 1: this is because P is a stochastic matrix.

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Definition

A real $r * r$ matrix is a stochastic matrix if its elements are all non-negative and the sums of the elements in each row sums to 1.

Theorem

The product of two stochastic matrices is another stochastic matrix.

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Theorem

Let P be the transition matrix of a Markov chain. The i, jth entry $p^{(n)}$ of the matrix P^n gives the probability that the Markov chain, starting in state x_i, will be in state x_j after n steps.

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Theorem

Let P be the transition matrix of a Markov chain, and let u be the probability vector which represents the starting distribution. Then the probability that the chain is in state x_i after n steps is the ith entry in the vector

$$u^{(n)} = uP^n$$

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The weather on day 0 is known to be sunny. This is represented by a vector in which the "sunny" entry is 100%, and the "rainy" entry is 0%: $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$
The weather on day 1 can be predicted by:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

Thus, there is an 90% chance that day 1 will also be sunny.

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The weather on day 2 can be predicted in the same way:

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \mathbf{x}^{(0)} P^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

General rules for day n are:

$$\mathbf{x}^{(n)} = \mathbf{x}^{(n-1)} P$$

$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)} P^n$$

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Definition

A state x_i of a Markov chain is called absorbing if it is impossible to leave it (i.e., $p_{i,i} = 1$). A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

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Definition

In an absorbing Markov chain, the probability that the process will be absorbed is 1.

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Definition

A Markov chain is called an ergodic chain if it is possible to go from every state to every state (not necessarily in one move). Ergodic Markov chains are called irreducible.

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Definition

A Markov chain is called a regular chain if some power of the transition matrix has only positive elements.

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Theorem

Let P be the transition matrix for a regular chain. Then, as $n \to \infty$ the powers P_n approach a limiting matrix W with all rows the same vector w. The vector w is a strictly positive probability vector (i.e., the components are all positive and they sum to one).